Sec 1.5 1^{st}-Order Linear Eqs

Type $\dfrac{dy}{dx} + P(x)y = Q(x)$. ☆ No y^2, y'', $(y')^2$, $\underline{yy'}$ ... allowed

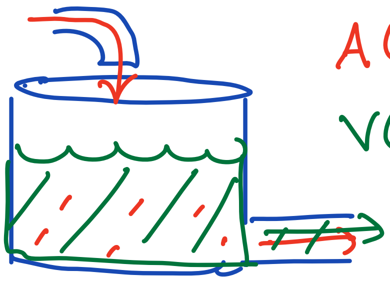
$\underline{\text{Ex}}$: • $y' + \frac{3}{x}y = e^x$ $\quad \left(P(x) = 3/x,\ Q(x) = e^x \right)$

• If $P(x) = 0$, then $\dfrac{dy}{dx} = Q(x)$ (separable)

• If $Q(x) = 0$, then $\dfrac{dy}{dx} = -P(x)y$ (separable)

Eg $y' = \dfrac{2x}{3y-5} \Rightarrow 3\boxed{yy'} - 5y' = 2x$

separable ... not linear



$A(t)$ = amt of chemical

$V(t)$ = volume of solution.

$$\frac{dA}{dt} + \frac{r_{out}}{V(t)} A(t) = r_{in} c_{in}$$

$\underline{\text{Ex}}$: Find a gen. sol. for ODE $y' + 2y = 7$ $\quad \left(\begin{array}{l} P(x) = 2 \\ Q(x) = 7 \end{array} \right)$

☆ Observe: $\dfrac{d}{dx}\left[\rho(x)\, y(x)\right] = \rho(x) y'(x) + \rho'(x) y(x)$.

and $\displaystyle\int \frac{d}{dx}[\rho \cdot y]\, dx = \rho(x) y(x) + C$

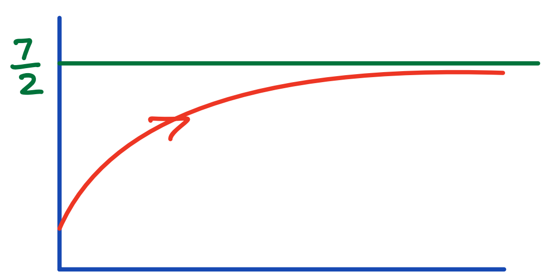
$\underline{\text{trick}}$: multiply by e^{2x}, $\quad y' + 2y = 7$

$\rightarrow \quad e^{2x} y' + 2e^{2x} y = 7e^{2x}$

$\Rightarrow \quad e^{2x} y' + (e^{2x})' y = 7e^{2x}$

$\Rightarrow \quad \displaystyle\int \frac{d}{dx}\left[e^{2x} y\right] dx = \int 7e^{2x}\, dx$

$\Rightarrow \quad e^{2x} y = \frac{7}{2} e^{2x} + C \Rightarrow \underline{y(x) = \frac{7}{2} + Ce^{-2x}}$



Working to "bridge" start and goal

Start with ODE $\{ y' + P \cdot y = Q \}$

If we can reach point $\int \frac{d}{dx}[\rho \cdot y]\,dx = \int \rho \cdot Q\,dx$,

then we get $\rho \cdot y = \int \rho \cdot Q\,dx + C$,

and can solve for $\boxed{y = \cdots,}$ so we have explicit solution.

So how do we get from $y' + Py = Q$

to $\int \frac{d}{dx}[\rho \cdot y]\,dx = \int \rho \cdot Q\,dx$?

Let's work "edges → middle":

Stage 1

Start	$y' + P \cdot y = Q$
↓	$(?)$
↑	
Goal	$\int \frac{d}{dx}[\rho \cdot y]\,dx = \int \rho \cdot Q\,dx$

Stage 2

Start	• $y' + P \cdot y = Q$
	• $\rho \cdot y' + \rho \cdot P \cdot y = \rho Q$
	$(?)$
	• $\frac{d}{dx}[\rho \cdot y] = \rho \cdot Q$
Goal	• $\int \frac{d}{dx}[\rho \cdot y]\,dx = \int \rho \cdot Q\,dx$

Stage 3

Start • $y' + P \cdot y = Q$

• $\rho \cdot y' + \rho \cdot P \cdot y = \rho Q$

connect? ✓ ? ✓

• $\rho \cdot y' + \rho' y = \rho Q$

• $\frac{d}{dx}[\rho \cdot y] = \rho \cdot Q$

Goal • $\int \frac{d}{dx}[\rho \cdot y]\,dx = \int \rho \cdot Q\,dx$

So last required connection is to make sure

$\{ \underline{\rho' = \rho \cdot P(x)} \}$ This is a "proscription" for ρ.

To fit, we need $\frac{d\rho}{dx} = \rho \cdot P(x)$ (separable! can solve for $\rho(x)$)

$$\int \frac{d\rho}{\rho} = \int P(x)\,dx$$

$$\ln|\rho| = \int P(x)\,dx$$

$$\Longrightarrow \rho = \pm e^{\int P(x)\,dx}$$

PICK "+", $\underline{\rho(x) = e^{\int P(x)\,dx}}$.

This connects $\{y' + Py = Q\}$ to our goal

$$\underline{\int \frac{d}{dx}[\rho \cdot y]\,dx = \int \rho \cdot Q\,dx}$$

Then, like we said, from here we get

$$\rho \cdot y = \int \rho \cdot Q\,dx + C$$

$$\Longrightarrow \boxed{y(x) = \frac{\int \rho(x) Q(x)\,dx + C}{\rho(x)}}$$

$\rho(x) = e^{\int P(x)\,dx}$ is called the <u>integrating factor</u>

<u>Ex</u> Find a gen. sol. for ODE $y' - y = 3e^{-x}$ $\quad P(x) = -1$, $\quad Q(x) = 3e^{-x}$

$$\rho(x) := e^{\int P(x)dx} = e^{\int -1\,dx} = e^{-x + k}$$

pick $k=0$

$\rho(x) := e^{-x}$ $\qquad e^{-x}y' - e^{-x}y = 3e^{-2x}$

$$\Rightarrow e^{-x}y' + (e^{-x})'y = 3e^{-2x}$$

$$\Rightarrow \frac{d}{dx}[e^{-x}y] = 3e^{-2x}$$

$$\Rightarrow e^{-x}y = \int (3e^{-2x})dx + C = -\frac{3}{2}e^{-2x} + C$$

$$\Rightarrow y(x) = -\frac{3}{2}e^{-x} + Ce^{x}$$

<u>Ex</u> Find a general solution for ODE

$$(x^2+1)y' + 3xy = 6x$$

$y' + Py = Q \longrightarrow P(x) = \ldots$ careful! Need to first put eq into <u>standard form</u> $y' + P(x)y = Q(x)$

$$\boxed{y' + \frac{3x}{x^2+1}y = \frac{6x}{x^2+1}} \longrightarrow y' + P\cdot y = Q(x)$$

$P(x) = \dfrac{3x}{x^2+1}$, $\qquad \rho(x) = e^{\int P(x)\,dx} = e^{\frac{3}{2}\ln|x^2+1| \cancel{+k}}$ $\quad$ pick $k=0$

$Q(x) = \dfrac{6x}{x^2+1}$, $\qquad = |\underbrace{x^2+1}_{>0 \text{ always}}|^{3/2} = (x^2+1)^{3/2}$

$$\rho y' + \rho' y = \rho Q \Rightarrow \frac{d}{dx}[\rho\cdot y] = \rho Q$$

$$\rightarrow \frac{d}{dx}\left((x^2+1)^{3/2}\cdot y\right) = (x^2+1)^{3/2}\cdot\frac{6x}{x^2+1}$$

$$\Rightarrow \quad (x^2+1)^{\frac{3}{2}} y = \int 6x(x^2+1)^{1/2}\,dx \quad + C$$

$$\Rightarrow \quad (x^2+1)^{3/2} y = 2(x^2+1)^{\frac{3}{2}} + C$$

$$\Rightarrow \quad \underline{y(x) = 2 + C(x^2+1)^{-3/2}}$$

Ex Find a gen. sol. for $\{xy' - 9y = x^{10}\cos(x)\}$, assuming $x > 0$*

Need $y' + P(x)y = Q(x)$

$$y' - \frac{9}{x}y = x^9\cos(x)$$

$P(x) = -9/x$

$Q(x) = x^9\cos(x)$

SET $\rho(x) = e^{\int P(x)\,dx} = e^{\int -9/x\,dx}$

$= e^{-9\ln|x| + k}$ (k picked $=0$)

$= |x|^{-9} \rightarrow x^{-9}$

(* b/c $x > 0$)

so $y' - \frac{9}{x}y = x^9\cos(x)$

$$\Rightarrow \quad x^{-9}y' - 9x^{-8}y = \cos(x)$$

$$\Rightarrow \quad \int \frac{d}{dx}\left[x^{-9}y\right]dx = \int \cos(x)\,dx$$

$$\Rightarrow \quad x^{-9}y = \sin(x) + C$$

$$\underline{y(x) = x^9\sin(x) + Cx^9}$$

General method: $y' + P\cdot y = Q$

$\rho\cdot y' + \rho\cdot P\cdot y = \rho\cdot Q$

Want to pick $\rho(x)$ in a way that makes

$$\rho\cdot y' + \underbrace{\rho\cdot P}\cdot y = \frac{d}{dx}[\rho\cdot y] \left(= \rho\cdot y' + \underbrace{\rho'}\cdot y\right)$$

$\Rightarrow$ Want $\rho' = \rho\cdot P \Rightarrow \frac{d\rho}{dx} = \rho\cdot P \Rightarrow \int \frac{d\rho}{\rho} = \int P\,dx$

$\Rightarrow \ln|\rho| = \int P(x)\,dx$

$\Rightarrow \rho = \pm e^{\int P(x)\,dx}$

Free to $\boxed{\text{pick } \rho(x) := e^{\int P(x)\,dx}}$.

Check: $\rho' = \left(e^{\int P(x)dx}\right)' = \left(e^{\int P(x)dx}\right)\cdot\left(\int P(x)dx\right)'$ ($\rho \cdot P$)

So: $y' + Py = Q \longrightarrow \rho y' + \rho\cdot P\cdot y = \rho\cdot Q$

$\Rightarrow \rho\cdot y' + \rho'\cdot y = \rho Q$

$\Rightarrow \int \frac{d}{dx}[\rho\cdot y]\,dx = \int \rho Q\,dx$

$\Rightarrow \rho\cdot y = \int \rho(x)\,Q(x)\,dx + C$

$$\Rightarrow \boxed{y(x) = \frac{\int \rho(x) Q(x)\,dx + C}{\rho(x)}}$$

$\rho(x) = e^{\int P(x)dx}$ is called the integrating factor

Ex Find a gen. sol. for ODE $y' - y = 3e^{-x}$

$P(x) = -1$

$Q(x) = 3e^{-x}$

$\rho(x) := e^{\int P(x)dx} = e^{\int -1\,dx} = e^{-x+k}$

pick $k=0$

$\rho(x) := e^{-x}$ $\quad e^{-x}y' - e^{-x}y = 3e^{-2x}$